July 15, 1948

Dear Dr. Dantzig,

I am very glad to learn that you are looking carefully into the procedure outlined in my paper on "A Numerical Method, etc." and I am much obliged for your sending me your comments.

Please forgive me now for writing to you more briefly than I would like to, and in a hurry. I am leaving tomorrow for the West, and you will imagine what pressure this produces. As you will see at the end of my letter, I am leaving two of your questions unanswered for this reason - I would have to go to some length to answer them adequately. I hope that we will be able to discuss these matters more fully when we meet in Madison on September 9th. I will, however, make another effort to get the time when I will have come to a relative state of equilibrium in a few weeks (in Los Alamos) and write to you again.

I will, therefore, only comment briefly on your remarks. My comments are as follows:

Page 3 - line 16 and last line: I wanted to refer here to footnote 2, but it might be good to refer to footnote 3 as well.

Page 3 - last line: I don't quite understand. Your formula and mine seem to be identical, except for a bracket. Do you suggest that that bracket is needed?

Page 5: You are right: (i) and (ii) are not exclusive of each other, but they are exclusive of (iii). I think that my text is in conformity with this, but it might be better to state this (that is, the possibility of (i) and (ii) both being true at the same time) explicitly as you suggest.

Page 16 - line 2: I think that the formula is correct as it stands. The right hand side has δ, the two terms on the left hand side should have the equivalents of δ after the insertion of the extra step. These are δ and δ', not δ' and $\tilde{\delta}$. The latter are the equivalents of $\tilde{\delta}$.

Page 17 - section 14: I had a procedure to show that one loses nothing by using a geometrical series at this point, I think, and I could probably reconstruct it. Do you want me to? I think, however, that it may be better to do this orally, perhaps when we meet in Madison.

Direct connection between your problem and the game - minimax problem:

I will try to write out a complete practical procedure in full detail. I hope that I can do this within a few weeks - at any rate before we meet in Madison.

I will also do this regarding your methodological questions. There is some connection between the two methods that I proposed, but it is not very rigid: Both are "descent" methods to minimise an expression which vanishes for a true solution. I had originally planned the first method for an analogy machine (presumably electrical, of the "linear equation solving circuit" type). The second method is, of course, intended for a digital machine. I am, therefore, most interested to learn that you will also have a digital calculation carried out with the first method - and I would very much like to know how it works out.

Apologizing for the incompleteness of this letter, and looking forward to seeing you in Madison, I am

Sincerely yours,

JOHN VON NEUMANN

JVN:LD

Dantzig

DEPARTMENT OF THE AIR FORCE
HEADQUARTERS UNITED STATES AIR FORCE
WASHINGTON

July 11, 1948

Professor John von Neumann
Institute of Advanced Study
Princeton, New Jersey

Dear Professor von Neumann:

I have been actively studying your paper, "A Numerical Method for Determination of a 0-Sum, 2-Person Game with Large Number of Strategies". I have some comments and some questions.

First some comments of an editorial nature.

	Suggested Change
Page 3 - line 16	(see also footnote 3)
Page 3 - last line	(see also footnote 3)
Page 3 - last line	$\kappa C + \lambda \succ (\kappa A + \lambda) IU$

Page 5 Alternates (i), (ii) and (iii) were confusing to me. It appears from what follows that either (iii) holds or (i) and/or (ii) holds.

Page 16 - unnumbered formula top of page should read

$$\frac{m}{\alpha'^2 \delta'^2} + \frac{m}{\alpha'^2 \delta''^2} \leq \frac{1}{\alpha^2 \delta^2}$$

where formula (37) becomes

$$\sqrt{2(1+\alpha)} - 1 > \alpha \sqrt{\frac{2}{1+\alpha} \cdot \frac{3+\alpha}{1+\alpha}}$$

and formula (38) becomes

$$\alpha < .32$$

These effect several other formulae.

Page 17 - Section 14. I cannot verify that it is best to choose a geometrical sequence.

In applying the technique to any large example of maximizing a linear function subject to linear restrictions, it is necessary to modify the technique. Thus it is easily verified that the min-max theorem for matrices is easily reducable to the above form but the converse does not follow. My main difficulty in making this modification (and this has been plaguing me) is that I do not have a geometrical picture of what is going on. Thus, I follow step by step the procedure but that is all. For example, I do not see why the ultimate X and U of the solution satisfy $X = \mathrm{Max}\,(A_i U, 0)$. I also do not see the connection between this procedure and the one outlined by you several months ago which I have written to you about. We have incidently, formalized the latter into a working procedure and Jack Laderman is trying it out on the nutrition problem.

One of the chief draw backs I find for the procedure, is the high price that is paid for obtaining precision. Actually after a while the same points are involved with only slight corrections on their weights. I have been working on the possibility of merging the two techniques (mine using n-points at a time, and yours using all points) to obtain an exact solution.

Yours sincerely,

George B. Dantzig

GEORGE B. DANTZIG
Mathematical Advisor

Dr von Neumann - I thought you might [be] interested in this correspondence with Prof. Tucker.
George B. Dantzig

Comptroller
20 June 1948

Professor A. W. Tucker
Department of Mathematics
Fine Hall
Princeton, New Jersey

Dear Professor Tucker:

The purpose of this letter is to suggest a problem that you might work on this summer in connection with our project that will be very valuable to us.

Professor John von Neumann has recently proposed a procedure in his paper " A Numerical Method for Determination of the Value and the Best Strategies of a 0-Sum, 2-Person Game with Large Numbers of Strategies". This procedure can be used to solve our problem of the maximization of a linear function of non negative variables subject to linear restrictions. Our primary concern, however, is in th application of the method to a "dynamic economy" -- see page I-9 fromula (15) of my paper on Programming in a Linear Structure. You will note that the fundamental matrix in this case is composed of zero elements except for blocks of sub matrices along the diagonal and just off the diagonal. Now it is important in any application that advantage of these zeros be taken. In general the number of multiplications is of the order $K.M.N^2$ (where K may range from 2 to as high as log N), M = no. of columns, N = no. of rows. If t = no. of time periods, M = tM', N = tN' where M' and N' are the number of activities and equipment items in a time period, the number of multiplications becomes $K.M'\,N'^2.t^3$. Indications are that my technique in the Linear Structure paper can be reduced from t^3 to t^2; while Professor von Neumann's proposed "convex-body" method by good luck can be reduced from t^3 to t.

What we would like is a well defined computational procedure that would be directly applicable to a small scale dynamic model. Would you like to work on this?

Yours sincerely

GEORGE B. DANTZIG

P.S. Copies of von Neumann's paper probably can be obtained from his office without trouble as it is mimeographed. I will be in California during July. Please correspond with Emil D. Schell who is guiding the mathematical work during my absence.

USAF Comptroller

June 18, 1948

Dear Prof von Neumann,

A proof that the "error" in the convex - body method is bounded by $c \cdot \lambda^n$, $0<\lambda<1$ is attached.

It should be noted that the problem stated in (1) is slightly more general then you originally formulated it in that Σx_i's (the weights assigned to points) are not required to be unity. Also not any elgible point P_i is selected to improve a solution* but the one stated in (12).

Sincerely yours,

George B. Dantzig

* i.e. an approximate solution

Given in n-space points R; $P_1, P_2 \cdots, P_m$, the problem is to determine $x_i \geq 0$ such that

$$(1) \quad R = \sum_1^m x_i P_i$$

Let $A = \sum_1^m v_i P_i$, $v_i \geq 0$ be any approximation to R. It is assumed R & P are normalized:

$$R^2 = 1 \text{ and } P_i^2 = 1$$

and also A is so chosen that $[R - kA]^2 = \text{Min}$ for $k = 1$. Setting

$$E = R - A,$$

A and E satisfy the condition $(A, E) = 0$. The next approximation

$$(2) \quad A' = x P_i + y A \qquad , (x > 0, y > 0)$$

is obtained by minimizing $[R - A']^2 = (E')^2$

$$(3) \quad (E')^2 = \underset{x,y}{\text{Min}} \; [R - x P_i - y A]^2$$

obtaining

$$(4) \quad x = \frac{(E, P_i)(A, A)}{(A, A) - (A, P_i)^2} \quad ; \quad 1 - y = \frac{(A, P_i)(E, P_i)}{(A, A) - (A P_i)^2}$$

$$(5) \quad (E')^2 = E^2 - \frac{(E, P_i)(A, A)}{(A, A) - (A, P_i)^2}$$

In order that there exist a solution to (1), it is necessary that there exist at least one P_i such that $(E, P_i) > 0$ for any $E = R - A$ where $A = \Sigma v_i P_i$, $(v_i \geq 0)$. Properties of the function (E, P_i) will now be investigated.

The set of all $A = \sum_1^m u_i P_i$, $(u_i \geq 0)$, $(A \cdot E = 0)$ defines a set S of points $E^* = E/|E|$ on the unit sphere, and a function

$$(6) \quad f(E^*) = \max_i (E^*, P_i) > 0 \quad ; (A \neq R, \text{ i.e. } E \neq 0)$$

The essential property of this function is

$$(7) \quad f(E^*) \geq b > 0$$

To prove this it will be assumed (for purposes of reaching a contradiction) g.l.b $f(E^*) = 0$. This implies simultaneous existence of three infinite sequences, (A_i), (E_i), and (E_i^*) such that

$$(8) \quad \lim A_i = A_0$$

$$(9) \quad \lim E_i = E_0$$

$$(10) \quad \lim E_i^* = \bar{E}_0$$

$$(11) \quad \lim f(E_i^*) = f(\bar{E}_0) = 0 \quad .$$

It is not difficult to show $A_0 = \sum_1^m X_i P_i$, $X_i \geq 0$, $(A_0 \cdot E_0) = 0$; so that if $A_0 \neq R$ then $E_0^* = \bar{E}_0$ and $f(\bar{E}_0) > 0$ contradicting (11). The possibility $A_0 = R$, $E_0 = R - A_0 = 0$ remains. From the conditions $(E_i^*, A_i) = 0$ for all i, it follows by continuity $(\bar{E}_0, A_0) = (\bar{E}_0, R) = 0$. Also $(\bar{E}_0, P_i) \leq 0$ for all P_i follows from (6) and (11). Thus, if a plane π is defined normal to the direction $\bar{E}_0$ <u>at the origin</u>, <u>all points P_i lie one side of</u> π. From $(\bar{E}_0, R) = 0$ it follows <u>π passes through</u> R. This situation is impossible if R is an <u>interior</u> point of the convex and demonstrates (7) holds;

except where R is exterior to or on the boundary of the convex.

If now P_i in (2) is selected

$$(12) \quad (E^*, P_i) = \operatorname*{Max}_j \, (E^*, P_j) > b > 0$$

it can be shown $x > 0$ and $y > 0$ — $x > 0$ follows from (4), but $y > 0$ requires a special proof* omitted here.

The relative improvement of the approximation A' over A is given by

$$\frac{E'^2}{E^2} = 1 - \frac{(E^*, P_i)^2}{1 - [(A_0, P_i)^2/(A,A)^2]} \leq 1 - (E^*, P_i)^2 \leq 1 - b^2$$

Setting $\lambda = 1 - b^2$, $0 < \lambda < 1$, the error E_n^2 (of the n^{th} approximation A_n) satisfies

$$E_n^2 \leq E_0^2 \cdot \lambda^n$$

* This proof assumes $(A-R)^2 \leq \operatorname*{Min}_{i, k \geq 0} (kP_i - R)^2$. The first approximation A_1 can always be selected to satisfy this property.

June 15, 1948

Dear Dr. Dantzig,

I was away from Princeton for several days and on my return I found your two recent letters. I am sending with the same mail copies of my paper, "A Numerical Method......" to Professor Barankin and to yourself, as requested.

I am very glad to see that you are interested in the method of this paper, and I am looking forward to hearing more from you about it. I am also very anxious to learn more about your estimate of the "error" of the convex-body method - that is, that you can improve it from $n^{-1/2}$ to λ^n, $0 < \lambda < 1$. An estimate of the latter type with any reasonable λ, $0 < \lambda < 1$, would obviously improve things very considerably. Can you give me an indication of your proof? What kind of conclusions can you draw concerning λ?

Sincerely yours,

JOHN VON NEUMANN

JVN:LD

Dr. George B. Dantzig
USAF - Comptroller
Pentagon Building
Washington 25, D. C.

June 4, 1948

Dear Dr. Dantzig,

I have just received your letter concerning the "convex-body method" in determining optimum strategies. I would be very glad if you would go ahead and try that method on a largish practical case. I need not tell you that I am also quite anxious to know how it works in real life.

No. 29 on unpubl. list B. H.

I suppose that you have received the mimeographed copy of "A Numerical Method for Determination of the Value and the Best Strategies of a 0-Sum, 2-Person Game with Large Numbers of Strategies," which I sent you a few days ago. The numerical method described there is actually a "constructivization" of the convex-body method, with a few additional viewpoints worked in: It is arranged in such a manner that the "alternative" which occurs in the book of Morgenstern and myself no longer appears explicitly. In connection with this, I have twisted the method around in such a manner that it assumes the shape of a double induction which gives the result to any desired degree of precision. I would be very much interested to have your comments and criticisms on the contents of that paper, and I would be very glad if you could take the time to look it over. You may find that the method described there is workable in the actual numerical experiment that you want to perform.

I am looking forward to learn your reactions about this matter. I am also very glad to learn that you will be able to attend the Iowa meeting, which promises to be interesting.

Sincerely yours,

JOHN VON NEUMANN

JVN:LD

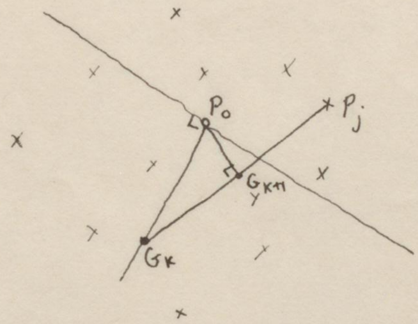
Dr. George B. Dantzig
USAF Comptroller
Pentagon Building
Washington 25, D. C.

USAF - Comptroller
Pentagon Bldg
Wash 25, D.C.
May 3, 1948

Rec'd June 3-1948

Dear Prof von Neumann,

Do you recall a rather apologetic suggestion you made at the very brief meeting (the one we were supposed to had had with Koopmans)?



If you have no objections, we would like very much to try the method out. It appears to have certain advantages for finding a feasible solutions to the linear inequality problem particularly in the dynamic case. As an initial test I have been thinking of using the nutrition problem again.

Thank you very much for suggesting my name as a participant at the Iowa meeting.

Yours sincerely

George B. Dantzig

P.S. We will go ahead with the above idea unless we hear otherwise from you.

$9^2 \times 77 = [illegible]237$

$15,315 : 6,237$

2.45

DEPARTMENT OF THE AIR FORCE
HEADQUARTERS UNITED STATES AIR FORCE
WASHINGTON

28 April 1948

Prof. von Neumann
The Institute for Advanced Study
Princeton, New Jersey

Dear Prof von Neumann:

At out last meeting you asked if I could supply you with the number of steps and the time required to solve the minimum cost diet problem by out present procedure.

You will recall that 77 foods and 9 nutrient elements were involved in this problem. The number of operations by type are as follows:

Type of Operations	No. of repetitions
Multiplication	15,315
Division	1,234
Addition of two numbers	14,561
Addition of 77 numbers	190
Addition of 9 numbers	85

To perform these computations with desk machines required 5 computers for 21 days, with 4 hours per day supervision by a mathematician.

The computational work was performed at the New York Mathematical Tables Project under the supervision of Jack Laderman.

We are presently trying the test problem on IBM equipment in our own machine installation, and I shall let you know how the timing compares with a hand job.

Sincerely,

George B. Dantzig

George B. Dantzig

252
504
3024

$5 \times 21 \times 8 = 840$ Mh

$840 \times 3600 = 3,024,000$ M sec

$\frac{3,024,000}{15,315} = 197$ M sec/mult

$\frac{197}{10} = 19.7$ eff.

+ 14,561
190·76 14,440
85·8 680
29,681 : 15,315
1.94

÷ 1,234 : 15,315
.08

+, ×, ÷ ~ 1.94, 1, .08

28 April 1948

Prof. von Neumann
The Institute for Advanced Study
Princeton, New Jersey

Dear Prof von Neumann:

At out last meeting you asked if I could supply you with the number of steps and the time required to solve the minimum cost diet problem by out present procedure.

You will recall that 77 foods and 9 nutrient elements were involved in this problem. The number of operations by type are as follows:

Type of Operations	No. of repetitions
Multiplication	15,315
Division	1,234
Addition of two numbers	14,561
Addition of 77 numbers	190
Addition of 9 numbers	85

To perform these computations with desk machines required 5 computers for 21 days, with 4 hours per day supervision by a mathematician.

The computational work was performed at the New York Mathematical Tables Project under the supervision of Jack Laderman.

We are presently trying the test problem on IBM equipment in our own machine installation, and I shall let you know how the timing compares with a hand job.

Sincerely,

George B. Dantzig

Dantzig

\- OUTLINE -

Selection of Optimum Military Programs

BOOK I

HIGH SPEED COMPUTATION OF PROGRAMS - A PROSPECTUS

BOOK II

MATHEMATICAL FORMULATION OF THE PROGRAM SELECTION PROBLEM

2nd Draft
USAF Comptroller
6 April 1948

BOOK I

High Speed Computation of Military Programs - A Prospectus

1. Purpose of this book.

2. Introduction

   2.1 Outline of need for rapid computation of programs.

   2.2 Outline of rapid selection of "best programs" by electronic computer.

   2.3 Outline of how inter-industry relations affects military programs and conversely.

   2.4 Outline of the role "Military Worth" plays in the selection of programs.

   2.5 Discussion of certain benefits which will accrue if the Electronic Program Computer project succeeds.

3. Historical Background of the Input-Output technique for studying Inter-Industry Relations.

   3.1 Work of Leontief.

   3.2 Work of BLS.

   3.3 Work of Von Neumann

   3.4 Work of Koopmans and others.

4. The Use of Input and Output Coefficients to express Air Force Structural Relationships.

   4.1 General Discussion of Terminology and Assumptions.

      4.11 Definition of Activity and Equipment Items.

      4.12 Examples of Different Types of Activities.

      4.13 Definition of term "Program".

      4.14 Definition of a basic time period for an Activity.

4.15 Assumption that equipment can be allocated to activities in a mutually exclusive way.

4.16 Assumption that activities are complete within a time period.

4.17 Assumption that activities are complete over time.

4.18 Assumption that Input Coefficient is Constant.

4.19 Assumption that Output Coefficient is Constant.

4.2 Training and Support Activities.

4.3 Combat and Support Activities.

4.4 Expressing Special Side Relations.

4.4.1 Conditions Restricting Level of an activity.

4.4.2 Conditions Expressing build-up of experience.

4.4.3 Conditions that Interrelate activities that occur in the same time period.

5. Selection of "Best" Programs or the application to General Strategy.

5.1 The General Problem of Selection of Programs.

5.11 Effect of Limitation of Resources.

5.12 Definition of a "Best" Program.

5.2 Dependence of Air Force Program on other programs both military and civilian.

5.3 Possibility of studying "Military Worth" by means of input-output relations of a Potential Enemy and Allied Economies.

5.3.1 Application to Offensive Strategy.

5.3.2 Application to Defensive Strategy.

5.4 Possibility of Extension to Min-Max Strategy.

6. The General-Purpose High Speed Digital Calculator and its use in Planning.

6.1 General Description and Historical Background of the machine.

6.2 Interest of other agencies in computers.

6.3 Operating Characteristics.

6.4 Cost.

6.5 Application to Selection of Optimum Program.

6.6 Application to Generation of Detailed Programs.

6.7 Statistical and Other Uses of the Machine.

7. Development of Factors

7.1 The magnitude of the problem of collecting and evaluation input and output coefficients.

7.2 The use of aggregate activities, equipment items, and time periods to develop broad programs.

7.2.1 Possibility of obtaining useful results quickly.

7.2.2 A method of testing new mathematical Procedures.

7.3 Kinds of Data Required in Different Areas

7.3.1 The Air Force

7.3.1.1 The Combat Model

7.3.1.2 The Training Model

7.3.1.3 Support Activities

7.3.1.4 Supply, Procurement, Storage, and Shipment Activities

7.3.2 The National Economy

7.3.2.1 Inter-Industry and Product Classifications

7.3.2.2 Classifying Military and Civilian Requirements on a common basis

7.3.3 Other Services

7.3.4 Allied Countries

7.3.5 Potential Enemy Countries

8. Recommended Actions by Different Agencies

8.1 National Security Resources Board

8.2 Research and Development

8.3 Munition Board

8.4 Other Civilian Agencies

8.5 Universities

8.6 Possibility of Integrated Planning by the Three Services

8.7 Air Force

8.7.1 Rand

8.7.2 Intelligence

8.7.3 A-4 and Wright Field

8.7.4 Training and Operations

8.7.5 Comptroller

8.7.5.1 Analysis and Evaluation

8.7.5.2 Maintenance of Library of Factors

8.7.5.3 Servicing Hqs. with Factors and Flexible Planning Guides

8.7.5.4 Servicing Staff with Optimum Programs

8.7.5.5 Preparation of Instructions by Mathematical Branch

8.8 Bureau of Standards

BOOK II

The Mathematical Formulation of the Problem
Programming in a Linear Structure

20.1 A Mathematical Model of an Economy

20.11 Introduction.

20.12 General Definitions and Concepts.

20.13 Special Assumptions on Structure.

20.14 Discussion of Assumptions.

20.15 Equations of a Dynamic System.

20.16 The Maximizing Function

20.2 A Procedure for Maximizing a Linear Function Subject to Linear Inequalities.

20.21 Introduction

20.22 Notation

20.23 Formulation

20.24 Pròperties of Feasible and Maximum Feasible Solutions.

20.25 Construction of a Feasible Solution.

20.26 Construction of a Maximum Feasible Solution.

20.27 Geometrical Interpretation of the Procedure.

20.3 Illustration of Computional Procedures

20.31 Arrangement of Work.

20.32 Example of Minimum Diet Problem.

20.33 Discussion of Technique.

20.4 Discussion of other Computional Techniques.

20.41 Prof. Von Neumanns paper on Maximizing a Linear Function Subject to Linear Restrictions.

20.42 Second Proposal (Tentative) of Prof. Von Neumann.

20.43 Proposal of Modifying equations so that they are non-linear (Jerry Cornfield)

20.5 Possibility of using Analog Methods

20.6 Discussion of Machine Characteristics and its Relation to proposed Mathematical Procedures.

20.7 Discussion of Unsolved Mathematical Problems.

20.7.01 Devise a procedure of passing from the maximum feasible solution of one problem to a maximum feasible solution of another problem.

20.7.02 If no exact feasible solution exists, can a "nearly feasible" solution be constructed.

20.7.03 Investigate the variation in the number of feasible solutions in general and for a "random" set of points.

20.7.04 Study nature of propagation of error in the algorithms used.

20.7.05 Devise techniques by which a broad program based on aggregated activities and equipment may be expanded in detail.

20.7.06 Investigate feasibility of eliminating "positive-linearly" dependent points.

20.7.07 Can blocks of zeros in the dynamic matrix be taken advantage of?

20.7.08 Generalize technique of two rotations in 3 space to generate feasible solutions in n-space.

20.7.09 Study various ways to modify a feasible solution based on all points in a complex to arrive at the maximum feasible solution.

20.7.10 Devise a technique for selecting a second point belonging to a feasible solution.

20.7.11 Can linear equations be modified to non linear equations to arrive quickly at an approximate solution?

20.7.12 Develop J.von Neumann's suggestion of modifying a weighted set of points.

20.7.13 Investigate special (not general) problems of non-linearity.

20.7.14 Devise procedures for solving a min-max strategy in a competitive linear economy.

20.7.15 What is the expected number of steps for presently used algorithms if the points are "random"?

20.7.16 Develop criteria by which points close together may be clustered until an approximate solution is obtained.

Office of Air Comptroller, AFAPA,
USAF - Pentagon Building,
Washington 25, D. C.
13 November 1947

Professor J. Von Neumann
Institute of Advanced Study
Princeton, N. J.

Dear Professor Von Neumann:

Inclosed is a copy of a procedure for obtaining a maximizing solution. Since it represents our best efforts to date, it may be of interest to you to know how far we have gotten.

There is a possibility that Colonel Prescott M. Spicer, Chief of the Program Analysis Division may also attend. He, like Marshall K. Wood, whom I spoke to you about on the telephone has been instrumental in setting up the Air Force Project. Both are keenly interested in your analysis of the problem even though they don't expect to follow any technical discussion. If Colonel Spicer does attend, we will fly up and land at a nearby field.

Sincerely yours,

George B. Dantzig

1 Incl.

GEORGE B. DANTZIG

Dantzig

RECOMMENDED PROCEDURE FOR FINDING FEASIBLE AND

MAXIMUM FEASIBLE SOLUTIONS

Let S be a finite set of points in Euclidean n-Space (R_n), - the coordinates of a point being given by (x_1, x_2, ...x_{n-1}; z). Let G be a fixed line parallel to the z axis, i.e., the line G is defined by ($x_1 = g_1$, $x_2 = g_2$, ...$x_{n-1} = g_{n-1}$). The projection parallel to the z axis of the points of set S on to the plane z=o, will be denoted by the set S'; the line G under this projection becomes a fixed point on the plane z=o, which we will denote by the same letter G.

The general problem is to assign to each point of S a <u>non-negative</u> weight, such that (1) the center of gravity, Q, of the system of weighted points lies on the line G and (2) the z coordinate of Q is maximum. A set of weights satisfying (1) will be referred to as a <u>feasible solution</u>. If it satisfies (2) it will be called a <u>maximum feasible solution</u>.

The set of all possible centers of gravity Q generated by assigning arbitrary non-negative weights to points of S forms a convex set of points bounded by a generalized polyhedral surface. The line G, if it cuts the convex at all, cuts it in two points corresponding to the minimum and the maximum solutions. Denote by Q the latter point, i.e., the point of the convex with maximum z coordinate on the line G. There exists n- points, P_i, on the polyhedral surface of S and a non-negative weighting of these n- points, such that the center of gravity of the n- point is Q. Considering now the set S as a whole, if P_i is assigned these weights and all other points are given the weight <u>zero</u>, then Q will be also center of gravity of the whole system and these weights will constitute the maximum solution. This solution in general is unique. It is the purpose of this report to give a constructive procedure for obtaining the P_i or showing that no feasible solution exists. The following two problems accordingly will be considered:

PROBLEM I:

From the set of points S', obtained from the projection of S and line G on the plane $z=o$, construct an n-1 dimensional simplex containing the point G as an interior point.

PROBLEM II:

Assuming the existence in R_n of at least one n-1 dimensional simplex determined by n- points P_i, ($P_i \subset S$), such that the line G intersects the hyper-plane defined by the vertices of the simplex in an interior point of the simplex, Q, construct a similar simplex such that Q has a maximum z coordinate.

PROCEDURE - PROBLEM I:

In this section the space under consideration is the R_{n-1} Space, $z=o$. Select arbitrarily n-1 points P_i, ($P_i \subset S'$). Let G_o be any fixed point in the plane of these points and interior to the (n-2) dim. simplex formed by these points, i.e.,

$$G_o = a_1 P_1 + a_2 P_2 \ldots + a_{n-1} P_{n-1},$$

where $a_i > 0$, $\sum_1^n a_i = 1$.

Let S_1' be the subset of points of S' on the same side of the plane determined by $P_1, P_2, \ldots P_{n-1}$, as the point G, i.e., all points $P \subset S'$ such that

$$\operatorname{Sgn}\left[P, P_1, \ldots P_{n-1}\right] = \operatorname{Sgn}\left[G, P_1, \ldots P_{n-1}\right],$$

where e.g.

$$\left[P, P_1, \ldots P_{n-1}\right] = \begin{vmatrix} x_1 & x_1^{(1)} & x_1^{(n-1)} \\ x_2 & x_2^{(1)} & x_2^{(n-1)} \\ x_{n-1} & x_{n-1}^{(1)} & x_{n-1}^{(n-1)} \\ 1 & 1 & 1 \end{vmatrix},$$

where the columns, except for the last row, are the coordinates in R_{n-1} Space of the corresponding points. In the event that S_1' is empty, no feasible solution exists, for, in this case, the center of gravity of the set of points S', whatever be their weighting, would have to lie on the side of the plane opposite G (or at best on the plane itself) contradicting the possibility of the point G becoming the center of gravity of S'.

If S_1' is not empty, let P be any point in S'. Form the n-1 dim. simplex $(P, P_1, P_2, \ldots P_{n-1})$. The line G_oG pierces two faces of this simplex in two points G_o and $G_1(P)$, where

$$G_1(P) = \lambda G_o + \mu G, \qquad (\lambda + \mu = 1).$$

For all $P \subset S_1'$, $\mu > 0$. If $\mu < 1$, then $G_1(P)$ lies between G_o and G. If $\mu > 1$, then G lies between G_o and $G_1(P)$. Since μ depends on the selection of P, let μ be <u>maximum</u> for $P = P_*$ and let

$$G_1 = G(P_*)$$

If $\mu > 1$ for P_*, then the points $(P_*, P_1, P_2, \ldots P_{n-1})$ form the desired simplex. If, on the other hand, $\mu < 1$, then replace the n-1 points $P_1, P_2, \ldots P_{n-1}$ by the vertices of that face of the simplex containing G_1. Denote by $(P_1^{(1)}, P_2^{(1)}, \ldots P_{n-1}^{(1)})$ the n-1 vertices of this face.

Repeat the above procedure with the new points $P_1^{(1)}, P_2^{(1)}, \ldots P_{n-1}^{(1)}$, using G_1 as new interior point, and obtain a new maximum μ. If $\mu > 1$, then the process will stop at this stage. If not, the process will continue through a <u>finite</u> number of steps, each generating points G_i with the order arrangement $G_o, G_1, G_2, \ldots G_k$, G on the line (G_oG). The last G_k and the points $P_1^{(k)}, P_2^{(k)}, \ldots P_{n-1}^{(k)}$ containing G_k have the property that either there exist no points of S' on the same side of the plane of these points as G or else $\mu > 1$. In the former case, no feasible solution exists; otherwise, $\mu > 1$, and the points $P_*^{(k)}, P_1^{(k)}, P_2^{(k)}, \ldots P_{n-1}^{(k)}$ form the desired simplex.

Proof:

$$G_k = b_1 P_1^{(k)} + b_2 P^{(k)} \ldots + b_{n-1} P_{n-1}^{(k)}, \qquad (b_i \geqq 0, \sum b_i = 1),$$

$$G_{k+1} = \cancel{a} c_1 P^{(k)} + \cancel{a} c_2 P_2^{(k)} \cdots + c_{n-1} P_{n-1}^{(k)}, \qquad (c_i \geqq 0, \sum c_i = 1),$$

where it has been arbitrarily assumed without loss of generality that G_oG pierces the face opposite to the vertex $P_1^{(k)}$, namely the face $(P_*^{(k)}, P_2^{(k)}, \ldots P_{n-1}^{(k)})$. From $\mu > 1$, it follows

$$G = \alpha G_k + \beta G_{k+1}, \qquad (\alpha \geqq 0, \alpha + \beta = 1);$$

whence substituting the values of G_k and G_{k+1} given above, it is easily observed that G may be expressed in the form

$$G = e_o P_*^{(k)} + e_1 P_1^{(k)} + \ldots e_{n-1} P_{n-1}^{(k)} .$$

and easily verified that $e_i \geqq 0$ and $\sum e_i = 1$.

PROCEDURE - PROBLEM II:*

In this section the space under consideration is R_n Space. Let $P_1^{(1)}, P_2^{(1)}, \ldots P_n^{(1)}$ be a feasible solution in R_n, i.e., if Q_1 is the intersection of the plane $(P_1^{(1)}, \ldots P_n^{(1)})$ with the line G, then $Q_1 = \sum a_i P_i^{(1)}$ where $a_i \geqq 0$, $\sum d_i = 1$. (A method of obtaining such a set of points by projecting points of S on the plane $z = o$, is provided by Problem I). Let S_1 be the set of points "above" this plane, i.e., points of S with z values greater than the value that would be obtained by substituting the first n coordinates $(x_1, \ldots x_n)$ of a point of S in the equation of the plane. If S_1 is not empty, let P be any point of S_1 and let G cut the faces of the simplex $(P, P_1, \ldots P_n)$ in the two points Q_1 and $Q_2(P)$. Consider the set of points $Q_2(P)$. Determine P such that $Q_2 = Q_2(P)$ has max z coordinate, and let $(P_1^{(2)}, P_2^{(2)}, \ldots, P_n^{(2)})$ be the vertices of that face of the simplex containing Q_2 as interior point. Repeat above process with $(P_1^{(2)}, P_2^{(2)}, \ldots, P_n^{(2)})$ in place of $(P_1^{(1)}, P_2^{(1)}, \ldots P_n^{(1)})$, form now the plane of these new points, and consider the set S_2 of points "above" this plane. The process will terminate at this stage if S_2 is empty. If not, form S_2, S_3, etc., the process terminating when an S_k is obtained

*Based on suggestions by Professor T. Koopmans, University of Chicago.

which is empty. In this manner a Q_k is obtained with maximum z coordinate. The process is finite because $Q_i \neq Q_j$, $(i \neq j)$ i.e., the z coordinate of Q_i forms a monotonically increasing sequence of values. Let $P_1^{(k)}, P_2^{(k)}, \ldots P_n^{(k)}$ be vertices of face containing Q_k.

The desired maximum feasible solution is obtained, of course, by assigning non-negative weights to the set of points $(P_1^{(k)}, P_2^{(k)}, \ldots P_n^{(k)})$ such that Q_k becomes their center of gravity and giving zero weight to all other points in S.

File

3 October 1947

MEMORANDUM FOR LIEUTENANT GENERAL E. W. RAWLINGS:

SUBJECT: Conference on the AF Electronic Program Computer with Professor J. Von Neumann at the Institute of Advanced Study, Princeton, 1 October 1947.

1. In a meeting arranged by the Bureau of Standards, the following problem was discussed with Professor Von Neumann:

"What computational procedures should be used in the determination of the best choice of AF activities, e.g. one which will either (a) maximize potential combat effort under a fixed budget or (b) minimize the budget under fixed objectives."

2. Professor Von Neumann indicated that this problem was typical of a large group of problems of considerable difficulty and general mathematical interest. He expressed a desire to give the problem serious thought for two or three weeks, after which he will contact the National Bureau of Standards and arrange further discussion of procedures to be used.

3. Professor Von Neumann is one of, if not the, leading U. S. mathematicians in this type of work and I feel we are exceptionally fortunate to be able to interest him in our problem.

GEORGE B. DANTZIG
Mathematician

cc: Professor J. Von Neumann
John Curtis
Ed Cannon
Albert Cahn
Marshall Wood

SAF - Comptroller
Pentagon Bldg.
Wash 25, D.C.

Dear Prof. von Neumann,

Thank you very much for your letter. I have just begun to study your paper on 0-sum 2-person games with large number of strategies. It goes without saying that I will give it intensive study and give you my reaction (I hope) next week.

With regard to your suggested "relaxation procedure" which I wrote to you about last time, I have set it up for computations. It appears to require about 10 arithmetic computations per point P_i per cycle. As soon as I am satisfied with the layout, I will turn it over to N.B.S to try out on a large problem.

The "error" of approximation after n corrections (you indicated once) was less than $k \cdot n^{-1/2}$. I believe I can show it to be less than $k' \cdot \lambda^n$, $0 < \lambda < 1$. The method of proof establishes only the existence of the λ but does not provide a satisfactory means for evaluating it.

Sincerely yours

George B. Dantzig